Large excursions of action within the resonance of a degenerate Hamiltonian system with two degrees of freedom

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We investigate the width of the resonance zone in a degenerate Hamiltonian system with two degrees of freedom, in which the Hamiltonian lacks the quadratic term in the Taylor expansion. This leads to larger excursions of action in the phase space than the nondegenerate one, and corresponding resonance frequency widths would become narrower. However, in contrast to the nonautonomous Hamiltonian system with one and half degree of freedom, we find that the above case is not generic and only occurs at particular resonances. An example relevant to the interaction of resonances is considered. Analytic results are verified in numerical simulations.

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I. INTRODUCTION

Many physical systems are conservative and motions could be described by the Hamiltonian system, such as oscillator behavior [1], dynamics of charged particle in magnetic fields [2], galactic motion [3], *n*-Body problem [4], plasma physics [5], Bose-Einstein condensate [6] and soliton theory [7] etc. If a Hamiltonian system is integrable, the phase space trajectories are regular (periodic or quasiperiodic) and lie on invariant tori. But most Hamiltonian systems are near integrable, i.e., they can be treated as perturbations of integrable systems, and exhibit stochastic or chaotic behavior. The KAM (Kolmogorov-Arnold-Moser) theorem [8–10] states that, under the nondegenerate and Diophantine conditions, most unperturbed invariant tori continue to exist in the perturbed Hamiltonian systems for a sufficient small perturbation.

In this paper, an alternative class of Hamiltonian systems which violate the nondegenerate condition is considered, and Rüssmann proved that the invariant tori can still exist with a weaker nondegeneracy [11]. The degenerate Hamiltonian systems have many novel properties in physical applications, including work on particle accelerators [12], plasma wave heating [13], fluid dynamics [14], and plasma stellerators [15]. In discrete dynamical system, the nontwist maps are just such degenerate systems. Howard and Hohs [16] and Howard and Humpherys [17] investigated a family of nonmonotonic radial twist maps. They found that these systems exhibit topological rearrangement of the invariant tori at some certain control parameter value. The same subject was studied by Del-Castillo-Negrete, Greene, and Morrison [18], who considered the periodic orbits and the transition to chaos in area- preserving nontwist maps. Moreover, Soskin and co-workers [19,20] showed that, within nonlinear resonance of the zero-dispersion Hamiltonian systems (i.e., the driving frequency is close to a multiple of the extremal eigenfrequency), the maximal variation of energy (the variable similar to action) is typically proportional to the perturbation amplitude ϵ to the power 1/3, and thus it is larger than that within the conventional resonance ($\sim\epsilon^{1/2}$). Recently, Rypina *et al.* [21] studied a nonautonomous Hamiltonian system with one and half degree of freedom and argued that the resonance frequency widths in the vicinity of degenerate resonant tori are generally narrower than those in the vicinity of nondegenerate resonant tori, which is beneficial to the stability of the motions near the degenerate resonant tori.

Investigation on the degenerate resonance and related dynamical properties in a Hamiltonian system with two or more degrees of freedom is more interesting since it contains richer dynamical behavior and is more important in physics, for instance, Chandre [22] constructed renormalization-group transformations in order to compute thresholds of break-up of KAM tori. In our work, by using a canonical transformation and average principle, we reduce an autonomous Hamiltonian system with two degrees of freedom into one degree of freedom through eliminating the resonant variable, then investigate the variation of action/frequency within the degenerate resonance. This paper is organized as follows. Section II is the analytical results. In Sec. III, we present an example to verify our analytic results. The conclusions and discussion are given in Sec. IV.

II. THEORETIC ANALYSIS

We consider a Hamiltonian system with two degrees of freedom,

$$H(\boldsymbol{I},\boldsymbol{\varphi}) = H_0(I_1,I_2) + \epsilon H_1(I_1,I_2,\varphi_1,\varphi_2), \qquad (1)$$

where $\boldsymbol{I}$ and $\boldsymbol{\varphi}$ are action-angular variables, ϵ is the small perturbation parameter. For the unperturbed system $H=H_0$, each torus is labeled by the action $I_i(i=1,2)$ and the corresponding frequency is $\omega_i=\partial H_0/\partial I_i(i=1,2)$.

A resonance occurs when $r\omega_1=s\omega_2$ at $\boldsymbol{I_0}=(I_{10},I_{20})$, where r and s are integers. We expand H_0 around the resonance point $\boldsymbol{I_0}$

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$$H_0(\boldsymbol{I}) = H_0(I_{10},I_{20}) + (\omega_{10},\omega_{20})\begin{pmatrix}\delta I_1\\ \delta I_2\end{pmatrix} + \frac{1}{2}(\delta I_1,\delta I_2)\omega'\begin{pmatrix}\delta I_1\\ \delta I_2\end{pmatrix} + \frac{1}{3!}(\delta I_1,\delta I_2)^2\omega''\begin{pmatrix}\delta I_1\\ \delta I_2\end{pmatrix} + \cdots, \tag{2}$$

where $\omega_{i0}=\frac{\partial H_0}{\partial I_i}\big|_{I_i=I_{i0}}$, $\delta I_i=I_i-I_{i0}$ $(i=1,2)$, and

$$\omega' = \left.\begin{pmatrix}\frac{\partial^2 H_0}{\partial I_1^2} & \frac{\partial^2 H_0}{\partial I_1\,\partial I_2}\\ \frac{\partial^2 H_0}{\partial I_1\,\partial I_2} & \frac{\partial^2 H_0}{\partial I_2^2}\end{pmatrix}\right|_{I_i=I_{i0}} = D, \tag{3}$$

$$\omega'' = \left.\left(\begin{pmatrix}\frac{\partial^3 H_0}{\partial I_1^3}\\ \frac{\partial^3 H_0}{\partial I_1^2\,\partial I_2}\\ \frac{\partial^3 H_0}{\partial I_1^2\,\partial I_2}\\ \frac{\partial^3 H_0}{\partial I_1\,\partial I_2^2}\end{pmatrix}\begin{pmatrix}\frac{\partial^3 H_0}{\partial I_1^2\,\partial I_2}\\ \frac{\partial^3 H_0}{\partial I_1\,\partial I_2^2}\\ \frac{\partial^3 H_0}{\partial I_1\,\partial I_2^2}\\ \frac{\partial^3 H_0}{\partial I_2^3}\end{pmatrix}\right)\right|_{I_i=I_{i0}} = \begin{pmatrix}L_1 & L_2\\ L_2 & L_3\end{pmatrix}. \tag{4}$$

If $\det(D)\neq 0$, then the system is nondegenerate, according to the KAM theorem most nonresonant invariant tori do not vanish but are only slightly deformed. Now, we focus on the degenerate case, i.e., $\det(D)=0$ and assume that any two of the vectors L_1, L_2, and L_3 are linearly independent. Under this weaker nondegenerate condition, Rüssmann proved the existence of invariant tori for a sufficient small perturbation [11].

To make a canonical transformation from the canonical variables $(\boldsymbol{I},\boldsymbol{\varphi})$ to $(\boldsymbol{J},\boldsymbol{\theta})$, we introduce the generating function [23]

$$F = (r\varphi_1 - s\varphi_2)J_1 + \varphi_2 J_2. \tag{5}$$

Then the following equations defines a canonical transformation:

$$I_1 = \frac{\partial F}{\partial\varphi_1} = rJ_1, \quad I_2 = \frac{\partial F}{\partial\varphi_2} = J_2 - sJ_1,$$

$$\theta_1 = \frac{\partial F}{\partial J_1} = r\varphi_1 - s\varphi_2, \quad \theta_2 = \frac{\partial F}{\partial J_2} = \varphi_2. \tag{6}$$

In the variables $\boldsymbol{J},\boldsymbol{\theta}$, the rate of change of the resonant slow variable $\dot\theta_1=r\dot\varphi_1-s\dot\varphi_2$ measures the slow deviation from resonance. Therefore, we can average the Hamiltonian $H(\boldsymbol{J},\boldsymbol{\theta})$ over the fast variable θ_2 to remove it

$$\bar H(\boldsymbol{J},\boldsymbol{\theta}) = \bar H_0(J_1,J_2) + \epsilon\bar H_1(J_1,J_2,\theta_1). \tag{7}$$

From Eq. (7), we have $\dot J_2=\frac{\partial\bar H}{\partial\theta_2}=0$, i.e., $J_2=J_{20}=\frac{s}{r}I_{10}+I_{20}$ $=$const.. Now, we have reduced the Hamiltonian (1) of two degrees of freedom into that of one degree of freedom

$$\bar H(J_1,\theta_1) = \bar H_0(J_1) + \epsilon\bar H_1(J_1,\theta_1), \tag{8}$$

which is an integrable system.

We expand the unperturbed part $\bar H_0(J_1)$ of the Hamiltonian (8) about $J_{10}(=I_{10}/r)$ in a Taylor series

$$\begin{aligned}\bar H_0(J_1) &= \bar H_0(J_{10}) + \left.\frac{\partial\bar H_0}{\partial J_1}\right|_{J_{10}}\delta J_1 + \frac{1}{2!}\left.\frac{\partial^2\bar H_0}{\partial J_1^2}\right|_{J_{10}}\delta J_1^2 + \frac{1}{3!}\left.\frac{\partial^3\bar H_0}{\partial J_1^3}\right|_{J_{10}}\delta J_1^3 + \frac{1}{4!}\left.\frac{\partial^4\bar H_0}{\partial J_1^4}\right|_{J_{10}}\delta J_1^4 + \mathcal{O}(\delta J_1^5),\\ &= \bar H_0(J_{10}) + b\,\delta J_1 + \frac{1}{2}c\,\delta J_1^2 + \frac{1}{6}d\,\delta J_1^3 + \frac{1}{24}e\,\delta J_1^4 + \mathcal{O}(\delta J_1^5),\end{aligned} \tag{9}$$

where the coefficients b, c, d, and e are as follows:

$$\begin{aligned}b &= \left.\frac{\partial\bar H_0(J_1)}{\partial J_1}\right|_{J_{10}} = \left.\left[\frac{\partial H_0(\boldsymbol{I})}{\partial I_1}\frac{\partial I_1}{\partial J_1} + \frac{\partial H_0(\boldsymbol{I})}{\partial I_2}\frac{\partial I_2}{\partial J_1}\right]\right|_{I_i=I_{i0}}\\ &= \omega_{10}r + \omega_{20}(-s) = 0,\end{aligned} \tag{10}$$

$$\begin{aligned}c &= \left.\frac{\partial^2\bar H_0(J_1)}{\partial J_1^2}\right|_{J_{10}}\\ &= \left.\left(r^2\frac{\partial^2 H_0(\boldsymbol{I})}{\partial I_1^2} - 2rs\frac{\partial^2 H_0(\boldsymbol{I})}{\partial I_1\,\partial I_2} + s^2\frac{\partial^2 H_0(\boldsymbol{I})}{\partial I_2^2}\right)\right|_{I_i=I_{i0}},\end{aligned} \tag{11}$$

$$\begin{aligned}d = \left.\frac{\partial^3\bar H_0(J_1)}{\partial J_1^3}\right|_{J_{10}} = &\left(r^3\frac{\partial^3 H_0(\boldsymbol{I})}{\partial I_1^3} - 3r^2 s\frac{\partial^3 H_0(\boldsymbol{I})}{\partial I_1^2\,\partial I_2}\right.\\ &\left.\left. + 3rs^2\frac{\partial^3 H_0(\boldsymbol{I})}{\partial I_1\,\partial I_2^2} - s^3\frac{\partial^3 H_0(\boldsymbol{I})}{\partial I_2^3}\right)\right|_{I_i=I_{i0}},\end{aligned} \tag{12}$$

$$\begin{aligned}e = \left.\frac{\partial^4\bar H_0(J_1)}{\partial J_1^4}\right|_{J_{10}} = &\left(r^4\frac{\partial^4 H_0(\boldsymbol{I})}{\partial I_1^4} - 4r^3 s\frac{\partial^4 H_0(\boldsymbol{I})}{\partial I_1^3\,\partial I_2}\right.\\ &+ 6r^2s^2\frac{\partial^4 H_0(\boldsymbol{I})}{\partial I_1^2\,\partial I_2^2} - 4rs^3\frac{\partial^4 H_0(\boldsymbol{I})}{\partial I_1\,\partial I_2^3}\\ &\left.\left. + s^4\frac{\partial^4 H_0(\boldsymbol{I})}{\partial I_2^4}\right)\right|_{I_i=I_{i0}}.\end{aligned} \tag{13}$$

Ignoring the constant term $\bar H_0(J_{10})$, the unperturbed part $\bar H_0(J_1)$ has the form

$$\bar H_0(J_1) = \frac{1}{2}c\,\delta J_1^2 + \frac{1}{6}d\,\delta J_1^3 + \frac{1}{24}e\,\delta J_1^4 + \mathcal{O}(\delta J_1^5). \tag{14}$$

Next, in order to get $\bar H_1$ in Eq. (8), we expand $H_1(\boldsymbol{I},\boldsymbol{\varphi})$ in Eq. (1) in a Fourier series

$$H_1(\boldsymbol{I},\varphi)=\sum_{l,m}H_{1,l,m}(\boldsymbol{I})\exp[i(l\varphi_1+m\varphi_2)], \tag{15}$$

where l and m are integers, and $H_{1,l,m}(\boldsymbol{I})$ is

$$H_{1,l,m}(\boldsymbol{I})=\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}H_1(\boldsymbol{I},\boldsymbol{\varphi})\exp[-i(l\varphi_1+m\varphi_2)]d\varphi_1 d\varphi_2. \tag{16}$$

Applying Eq. (6) to Eq. (15), the transformed H_1 is

$$H_1(\boldsymbol{J},\boldsymbol{\theta})=\sum_{l,m}H_{1,l,m}\exp\left\{i\left[\frac{l}{r}(\theta_1+s\theta_2)+m\theta_2\right]\right\}=\sum_{l,m}H_{1,l,m}\exp\left\{\frac{i}{r}[l\theta_1+(ls+mr)\theta_2]\right\}. \tag{17}$$

We average $H_1(\boldsymbol{J},\boldsymbol{\theta})$ over the fast variable θ_2 and get $\bar{H}_1(\boldsymbol{J},\boldsymbol{\theta})$

$$\bar{H}_1(\boldsymbol{J},\boldsymbol{\theta})=\frac{1}{2\pi}\int_0^{2\pi}\sum_{l,m}H_{1,l,m}\exp\left\{\frac{i}{r}[l\theta_1+(ls+mr)\theta_2]\right\}d\theta_2, \tag{18}$$

where only the terms of $ls+mr=0$ left. Let $\frac{l}{r}=-p$, then we have

$$\bar{H}_1(\boldsymbol{J},\theta_1)=\sum_{p=-\infty}^{\infty}H_{-pr,ps}(\boldsymbol{J})\exp(-ip\theta_1). \tag{19}$$

Since $\bar{H}_1(\boldsymbol{J},\theta_1)$ is independent of θ_2, the action $J_2=J_{20}=\text{const}$. The Fourier coefficients $H_{-pr,ps}$ generally fall off rapidly as $|p|$ increases, approximately we keep only $p=0,\pm 1$ terms just as [23] did, in this case the excursion of action should include the main effect of $\bar{H}_1(\boldsymbol{J},\theta_1)$. It is not difficult to extend $\bar{H}_1(\boldsymbol{J},\theta_1)$ to more Fourier modes, but this will not change our main results. Bearing in mind that the Hamiltonian is real, we have $H_{-r,s}=H_{r,-s}$ and the perturbation term becomes

$$\epsilon\bar{H}_1(J_1,\theta_1)=\epsilon\bar{H}_{0,0}(J_1)+2\epsilon\bar{H}_{r,-s}(J_1)\cos\theta_1. \tag{20}$$

We expand the terms $\bar{H}_{0,0}(J_1)$ and $\bar{H}_{r,-s}(J_1)$ in powers of J_1 at J_{10},

$$\epsilon\bar{H}_{0,0}(J_1)=\epsilon\bar{H}_{0,0}(J_{10})+\left.\frac{\partial\bar{H}_{0,0}}{\partial J_1}\right|_{J_1=J_{10}}(\epsilon\delta J_1)+\mathcal{O}(\epsilon\delta J_1^2), \tag{21}$$

$$\epsilon\bar{H}_{r,-s}(J_1)=\epsilon\bar{H}_{r,-s}(J_{10})+\left.\frac{\partial\bar{H}_{r,-s}}{\partial J_1}\right|_{J_1=J_{10}}(\epsilon\delta J_1)+\mathcal{O}(\epsilon\delta J_1^2). \tag{22}$$

Then, omitting the constant terms and keeping only terms of the lowest order in $\epsilon\delta J_1$, we finally obtain the reduced Hamiltonian describing the motion near the resonance

$$\bar{H}(\delta J_1,\theta_1)=\frac{1}{2}c\,\delta J_1^2+\frac{1}{6}d\,\delta J_1^3+\frac{1}{24}e\,\delta J_1^4+2\epsilon\bar{H}_{r,-s}(J_{10})\cos\theta_1. \tag{23}$$

We can see that the excursion of action J_1 depends on the coefficient c. If c does not vanish, the maximal excursion ΔJ_1 within the resonance is of order $\epsilon^{1/2}$, which is the same order as that of the nondegenerate case. When $c=0$, ΔJ_1 will possess lower order than $\epsilon^{1/2}$. However, the degeneracy condition alone cannot guarantee the lack of the quadratic term in the Eq. (23).

In essence, we are seeking a solution of the system of the three algebraic equations:

$$r/s=\omega_2/\omega_1 \quad \text{(resonance condition)},$$

$$\det(D)=0 \quad \text{(degeneracy condition)},$$

$$c=0. \tag{24}$$

For a given (rational) value of the quantity r/s, the system of three equations with just two variables (J_1 and J_2) typically does not have any solution. But solutions may still exist in some special cases, e.g., like that one considered in Sec. III.

In the following, suppose that the above system (24) is satisfied at the exact resonance point $\boldsymbol{I_0}$, the maximal excursion of action J_1 is given by half the separatrix width (at $\theta_1=0$),

$$\Delta J_1=\left|\frac{24\bar{H}_{r,-s}(J_{10})\epsilon}{d}\right|^{1/3}, \quad \text{if } d\neq 0, \tag{25}$$

which is larger than the conventional scale $\sim\epsilon^{1/2}$. This outcome is comparable to Soskin *et al.*'s derivations in the zero-dispersion nonlinear resonance [20].

With Eq. (25) for the maximal excursion of action within the degenerate resonance, we can calculate the corresponding resonance frequency width

$$\Delta\omega_{J_1}=\frac{1}{2}d\Delta J_1^2=|6\sqrt{2}d\bar{H}_{r,-s}(J_{10})\epsilon|^{2/3}. \tag{26}$$

The maximal resonance frequency width is estimated to be $\Delta\omega_{J_1}\sim\epsilon^{2/3}$, while for the nondegenerate case it is $\sim\epsilon^{1/2}$, thus the former is narrower when ϵ is small. In some Hamiltonian which we will show below, the coefficient d vanishes along with c, and the resonance frequency width could become even smaller.

III. EXAMPLE

We now illustrate the above results using a modified model of coupling resonance diffusion [24]. Assuming the Hamiltonian has the form

$$H(\boldsymbol{I},\boldsymbol{\varphi})=H_0(I_1,I_2)+\epsilon H_1(I_1,I_2,\varphi_1,\varphi_2),$$

$$H_0=A(I_1^{4/3}+I_2^{4/3}+\mu I_1 I_2),$$

$$H_1 = -2A^{1/2}I_1^{1/3}I_2^{1/3}\cos\varphi_1\cos\varphi_2, \quad (27)$$

where the coefficient A is a parameter, μ and ϵ are small parameters ($0<\mu,\epsilon\ll 1$). If the unperturbed part H_0 satisfies the degenerate condition, i.e.,

$$\det(D) = \left|\begin{matrix} \frac{\partial^2 H_0}{\partial I_1^2} & \frac{\partial^2 H_0}{\partial I_1\,\partial I_2} \\ \frac{\partial^2 H_0}{\partial I_1\,\partial I_2} & \frac{\partial^2 H_0}{\partial I_2^2}\end{matrix}\right|_{I_i=I_{i0}} = (4A/9)^2(I_{10}I_{20})^{-2/3} - A^2\mu^2 = 0, \quad (28)$$

then we have $I_{10}I_{20}=(\frac{4}{9\mu})^3$.

Substituting the expression of H_0 in Eq. (11) and let $c=0$, we obtain

$$2I_{10}^{-2/3}(r/s)^2 - 9\mu(r/s) + 2I_{20}^{-2/3} = 0. \quad (29)$$

Here, we simply take $I_{10}=I_{20}=(\frac{2}{3\sqrt{\mu}})^3$, then we have $r/s=1$. In addition, from Eqs. (11)–(13) the coefficients in Eq. (14) have also been determined to be $c=d=0$ and $e=\frac{5\cdot 3^4}{2^4}A\mu^4\neq 0$.

Next, let us check the weaker nondegenerate condition, now the vectors L_1, L_2 and L_3 in Eq. (4) are as follows:

$$L_1 = \left.\begin{pmatrix} \frac{\partial^3 H_0}{\partial I_1^3} \\ \frac{\partial^3 H_0}{\partial I_1^2\,\partial I_2}\end{pmatrix}\right|_{I_i=I_{i0}} = \begin{pmatrix} -\frac{8}{27}AI_{10}^{-5/3} \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{9}{4}A\mu^{5/2} \\ 0\end{pmatrix}, \quad (30)$$

$$L_2 = \left.\begin{pmatrix} \frac{\partial^3 H_0}{\partial I_1^2\,\partial I_2} \\ \frac{\partial^3 H_0}{\partial I_1\,\partial I_2^2}\end{pmatrix}\right|_{I_i=I_{i0}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (31)$$

$$L_3 = \left.\begin{pmatrix} \frac{\partial^3 H_0}{\partial I_1\,\partial I_2^2} \\ \frac{\partial^3 H_0}{\partial I_2^3}\end{pmatrix}\right|_{I_i=I_{i0}} = \begin{pmatrix} 0 \\ -\frac{8}{27}AI_{20}^{-5/3} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{9}{4}A\mu^{5/2}\end{pmatrix}. \quad (32)$$

It is obvious that L_1 and L_3 are linearly independent. According to Rüssmann's theorem [11], the KAM tori still exist.

Applying the canonical transformation (6) ($r=s=1$) to the Hamiltonian (27), we can rewrite the Hamiltonian in terms of new variables $\boldsymbol{J},\boldsymbol{\theta}$

$$H = A[J_1^{4/3} + (J_2 - J_1)^{4/3} + \mu J_1(J_2 - J_1)] - \epsilon(2A^{1/2})J_1^{1/3}(J_2 - J_1)^{1/3}\cos(\theta_1+\theta_2)\cos\theta_2. \quad (33)$$

Now proceeding as in Sec. II, by averaging Eq. (33) over the fast variable θ_2, we get

$$\bar{H} = \bar{H}_0 + \epsilon\bar{H}_1 = A[J_1^{4/3} + (J_2 - J_1)^{4/3} + \mu J_1(J_2 - J_1)] - \epsilon(A^{1/2})J_1^{1/3}(J_2 - J_1)^{1/3}\cos\theta_1. \quad (34)$$

As in Eq. (23), we expand the averaged Hamiltonian (34) about J_{10}, and obtain the reduced Hamiltonian with one degree of freedom,

$$\bar{H}(\delta J_1,\theta_1) = B\,\delta J_1^4 - \frac{4A^{1/2}}{9\mu}\epsilon\cos\theta_1, \quad (35)$$

where we take the constant $J_2=J_{20}$, the coefficient $B=\frac{e}{24}=\frac{5\cdot 3^3}{2^7}A\mu^4$.

Thus, the maximal δJ_1 excursion of the separatrix is

$$\Delta J_1 = \left|\frac{2\frac{4A^{1/2}}{9\mu}\epsilon}{B}\right|^{1/4} \approx 0.975\mu^{-5/4}\epsilon^{1/4} \quad (A\approx 0.87), \quad (36)$$

and the corresponding frequency width is

$$\Delta\omega_{J_1} = 4B\Delta J_1^3 \approx 3.4\mu^{1/4}\epsilon^{3/4}. \quad (37)$$

The scaling $\Delta\omega_{J_1}\sim\mu^{1/4}\epsilon^{3/4}$ shows that, when ϵ is small and $\mu<\mathcal{O}(\epsilon^{-1})$, resonance frequency widths in the vicinity of degenerate resonant tori are narrower than those ($\sim\epsilon^{1/2}$) in the vicinity of nondegenerate resonant tori.

Moreover, to certify the above analytical results by numerical simulation, we introduce a nondegenerate Hamiltonian H',

$$H'(\boldsymbol{I}',\boldsymbol{\varphi}') = H_0'(I_1',I_2') + \epsilon H_1'(I_1',I_2',\varphi_1',\varphi_2'),$$

$$H_0' = A(I_1'^{4/3} + I_2'^{4/3}),$$

$$H_1' = -2A^{1/2}I_1'^{1/3}I_2'^{1/3}\cos\varphi_1'\cos\varphi_2', \quad (38)$$

whose perturbation term $\epsilon H_1'$ is the same as ϵH_1 in Eq. (27), but H_0' is not. The unperturbed part H_0' differs from H_0 only in a small quantity of order μ ($0<\mu\ll 1$). We consider the Hamiltonian system H' around the same 1:1 resonance point $I_{10}=I_{20}=(\frac{2}{3\sqrt{\mu}})^3$ as in the system H. As we did above, H' can be transformed to

$$\bar{H}'(\delta J_1',\theta_1') = \frac{1}{2}c'\,\delta J_1'^2 - \frac{4A^{1/2}}{9\mu}\epsilon\cos\theta_1', \quad (39)$$

where $c'=2A\mu\neq 0$. Then we can obtain the resonance frequency width

$$\Delta\omega_{J_1'} \approx 1.7\epsilon^{1/2}. \quad (40)$$

The above two models are near integrable and the trajectories in the phase space are computed numerically. On the manifold of the Hamiltonian integral H=const., the motion takes place on a three-dimension subspace embedded in the four-dimension phase space of $(J_1,J_2,\theta_1,\theta_2)$. The usual two-dimensional surface of section (J_1,θ_1) is defined by setting $\theta_2=\pi$ and $\dot{\theta}_2>0$. Then we can map such surface of section (J_1,θ_1) into the surface (ω_{J_1},θ_1) through the relationship $\omega_{J_1}=\partial H_0/\partial J_1$. We produce such graphs of $\omega_{J_1}(\omega_{J_1'})$ versus

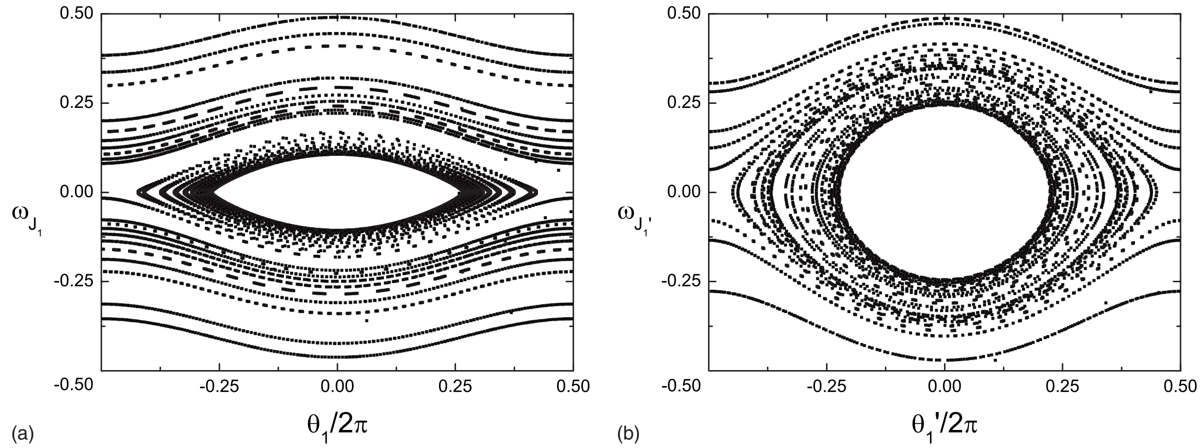

FIG. 1. Graphs of $\omega_{J_1}(\omega_{J_1'})$ versus $\theta_1(\theta_1')$ in surfaces of section $[\theta_2(\theta_2')=\pi, \dot{\theta}_2(\dot{\theta}_2')>0]$ for two similar Hamiltonian systems: (a) the degenerate case H; (b) the nondegenerate case H'. In both cases the small parameters are $\mu=\epsilon=0.05$.

$\theta_1(\theta_1')$ for the above two systems H and H' ($\mu=\epsilon=0.05$) in the neighborhood of $\boldsymbol{J_0}$ (see Fig. 1). The degenerate resonance width $\Delta\omega_{J_1}$ and the nondegenerate one $\Delta\omega_{J_1'}$ shown in Fig. 1 coincide well with theoretical estimation Eq. (37) ($\sim\epsilon^{3/4}$) and Eq. (40) ($\sim\epsilon^{1/2}$), respectively. This indicates that the foregoing analytical results are correct.

IV. CONCLUSION AND DISCUSSION

In summary, we have investigated the property of the resonance zone in a degenerate Hamiltonian system with two degrees of freedom. For the degenerate system, the Rüssmann weaker nondegenerate condition [11] guarantees the persistence of invariant tori for a sufficient small Hamiltonian perturbation. By using a canonical transformation and average principle, we reduce the degenerate Hamiltonian into one degree of freedom through eliminating the resonant variable. We have shown that maximal excursions of action Δ_{J_1} within the degenerate resonant tori depend on the coefficient c [Eq. (11)] of the quadratic term in the Taylor expansion over the action from the exact resonance in the reduce system. Suppose that c vanishes, maximal excursions of action are proportional to at least the order $\epsilon^{1/3}$, which is larger than the nondegenerate case. Besides, the corresponding resonance frequency widths would become narrower.

In the nonautonomous Hamiltonian systems with one and half degree of freedom, the contraction of resonance frequency widths can be observed near the degenerate resonant tori according to Rypina's argument [21]. However, this finding is not always valid for the Hamiltonian systems with two degrees of freedom because the degenerate condition alone cannot guarantee the vanishment of the coefficient c. It indicates that, in the Hamiltonian system with two degrees of freedom, the smaller resonance frequency widths can only occur when the system (24) have solution. Other than these particular cases, degenerate resonance widths are the same order as nondegenerate resonance widths, and we cannot get the above interesting results.

It is well known that resonance frequency widths are very important because resonance overlap could be responsible for the destruction of KAM tori [25,26]. The interaction of two adjoining resonances depends on the ratio of the sum of their maximal frequency widths $\Delta\omega=\Delta\omega_1+\Delta\omega_2$ to the frequency distance $\mathcal{D}$ between them. If $\frac{\Delta\omega}{\mathcal{D}}\geq 1$, the adjacent resonances would overlap and this implies the appearance of stochastic instability of motions. Considering the neighboring resonances at $\frac{\omega_{I_1}}{\omega_{I_2}}$ values of $\frac{s}{r}$ and $\frac{s}{r}+\frac{p}{q}$, we have $\frac{\omega_{J_1}}{\omega_{J_2}}=\frac{\omega_{I_1}}{\omega_{I_2}}\cdot r-s$, therefore these two resonances are separated by the distance $\mathcal{D}=\left|\frac{s}{r}-\frac{p}{q}\right|r\omega_{J_2}$ in the frequency ω_{J_1} region. Although ω_{J_2} is of order unit [23] compared with ϵ, the frequency distance $\mathcal{D}$ could be very small when two rational numbers $\frac{s}{r}$ and $\frac{p}{q}$ are sufficiently close. Besides, we should notice that $\Delta\omega$ is a quantity of $\beta\epsilon^{\alpha}$, and the coefficient β could be large. Therefore, for the small perturbation amplitude ($\epsilon\ll 1$), resonance frequency widths $\beta\epsilon^{\alpha}$ are probable to reach the scale of the distance in frequency between neighboring resonances, leading to the overlap of resonances.

Under the above situation, the small resonance frequency widths ($\alpha\geq 2/3$) near the degenerate resonant tori would make separatrices of two adjacent resonances have less possibility to touch each other, and the invariant tori between them would be more stable. On all accounts, this stability issue should be analyzed and calculated according to concrete physical systems, not only by an order-of-magnitude estimate of $\Delta\omega$ or $\mathcal{D}$. Our results would be helpful and applicable to the further study of some physical models that could be described by the degenerate Hamiltonian system. This is also the future task of our work.

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